APUNTES DEL CUADERNO DE CALCULO SUPERIOR

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TEMA: TÉCNICAS DE INTEGRACIÓN

INTEGRACIÓN DIRECTA MEDIANTE TEOREMAS

1. $\int x^3 dx = \frac{x^4}{4} + C$

2. $\int \frac{x^5}{5} + 2dx = \int \frac{x^5}{5} dx + \int 2dx = \frac{1}{5}\int x^5 dx + 2\int dx = \frac{1}{5}\left[\frac{x^6}{6} + C_1\right] + 2[x + C_2] = \frac{x^6}{30} + 2x + C$

3. $\int_0^5 3x^4 dx = 3\int_0^5 x^4 dx = 3\left[\frac{x^5}{5}\right]_0^5 = 3\left[\frac{5^5}{5} - \frac{0^5}{5}\right] = 3[5^4] = 1875$

4. $\int x^3\sqrt{x^4 + 11}\, dx = \int \frac{du}{4} u^{\frac{1}{2}} = \frac{1}{4}\int u^{\frac{1}{2}} du = \frac{1}{4}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] = \frac{1}{6}u^{\frac{3}{2}} = \frac{1}{6}(x^4 + 11)^{\frac{3}{2}} + C$

$$u = x^4 + 11 \qquad du = 4x^3 dx$$

5. $\int \frac{x^2 - x}{x+1} dx = \int \frac{x^2}{x+1} dx + \int \frac{x}{x+1} = \int \frac{u}{x+1}\left(\frac{du}{2x-1}\right) = \int \frac{x^2 - x}{u} du = \int \frac{(u-1)^2 - (u-1)}{u} du \rightarrow$
$\frac{x^2}{x+1} - \frac{x}{x+1}$

$$\int \frac{x^2}{x+1} dx - \int \frac{x}{x+1} dx \rightarrow u = x^2, \quad du = xdx$$

$$\int \left[\frac{2}{x+1} + x - 2\right] dx = \int \frac{2}{x+1} dx + \int xdx - 2\int dx = 2\int \frac{du}{u} + \int \frac{x^2}{2} - 2x + C_1$$

$$u = x + 1, \quad du = dx$$

$$2\ln|u| + \frac{x^2}{2} - 2x + C_1 = \mathbf{2\ln|x+1| + \frac{1}{2}x^2 - 2x + C}$$

SUSTITUCIONES SIMPLES

1. $\int (x-1)^4 dx = \int u^4 du = \frac{u^5}{5} + C = \frac{\boldsymbol{(x-1)^5}}{\mathbf{5}} + \boldsymbol{C}$

$$u = x-1, \quad du = dx$$

2. $\int \sqrt{2x}dx = \int u^{\frac{1}{2}}\frac{du}{2} = \frac{1}{2}\int u^{1/2}du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C = \frac{\mathbf{1}}{\mathbf{3}}(\boldsymbol{2x})^{\mathbf{3/2}} + \boldsymbol{C}$

$$u = 2x, \quad \frac{du}{2} = dx$$

3. $\int x(x^2+1)^4 dx = \int u^2(\frac{du}{2}) = \frac{1}{2}\int u^4 du = \frac{1}{2}\left[\frac{u^5}{5}\right] + C = \frac{\mathbf{1}}{\mathbf{10}}(\boldsymbol{x^2}+\mathbf{1})^{\mathbf{5}} + \boldsymbol{C}$

$$u = x^2+1, \quad \frac{du}{2} = xdx$$

4. $\int \frac{e^x}{1+2e^x}dx = \int (e^x dx)\left(\frac{1}{1+2e^x}\right) = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{ln}|\mathbf{1}+\mathbf{2}\boldsymbol{e^x}| + \boldsymbol{C}$

$$u = 1+2e^x, \quad \frac{du}{2} = e^x dx$$

5. $\int 3t\sqrt{2+t^2}dt = 3\int t\,(2+t^2)^{1/2}dt = \frac{3}{2}\int u^{1/2}du = \frac{3}{2}\left[\frac{u^{3/2}}{3/2}\right] + C = u^{3/2} + C =$ $(\mathbf{2}+\boldsymbol{t^2})^{\mathbf{3/2}} + \boldsymbol{C}$

$$u = 2+t^2, \quad \frac{du}{2} = t\,dt$$

INTEGRACIÓN POR PARTES

$$\int \Delta x[v(x)u(x)] = \int v'(x)u(x)dx + \int x(x)u'(x)$$

$$v(x)u(x) = \int v'(x)u(x)\,dx + \int v(x)u'(x)dx$$

$$\int v'(x)u(x)dx = v(x)u(x) - \int v(x)u'(x)dx$$

1. $\int xCos\ x\ dx$

$$u = x, \qquad dv = Cos\ x\ dx$$

$$du = dx, \qquad v = Sen\ x$$

$$\int x\ Cos\ x = x(sen\ x) - \int sen\ x\ dx = \boldsymbol{x\ Sen\ x} + \mathbf{cos}\ \boldsymbol{x} + \boldsymbol{C}$$

2. $\int_{\mathbf{1}}^{\mathbf{2}} \boldsymbol{ln\ x\ dx} = xlnx - \int x\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x]_1^2 = [2\ln|2| - 2] -$ $[\ln|1| - 1] \approx 0.39$

$$\mathrm{u = lnx}, \qquad \mathrm{dv = dx}$$

$$\mathrm{du} = \frac{1}{\mathrm{x}}\mathrm{dx}, \qquad \mathrm{v = x}$$

3. $\int e^x\, senx\, dx = e^x sen\, x - \int e^x Cos\, x\, dx$

$$u = xSenx, \qquad dv = e^x dx$$
$$du = Cos\, x\, dx, \qquad v = e^x$$
$$= e^x Sen\, x = \left[e^x Cosx - \int e^x(-senx)dx\right] = e^x Senx - \left[e^x Cosx + \int e^x senx\, dx\right]$$
$$\int e^x Senx\, dx = e^x Senx - e^x Cosx - \int e^x Senx\, dx$$
$$\int e^x Senxdx = \frac{1}{2} e^x Senx - \frac{1}{2} e^x Cosx + K$$

4. $\int Sec^3\theta d\theta = \int Sec^2\theta Sec\theta\, d\theta$

$$u = Sec\theta, \quad dv = Sec^2\theta$$
$$du = Sec\theta tan\theta, \qquad v = tan\theta$$
$$= tan\theta sec\theta - \int tan\theta\, Sec\theta tan\theta d\theta$$
$$- \int tan^2\theta Sec\theta d\theta$$
$$- \int Sec\theta(Sec^2\theta - 1)d\theta$$
$$- \int Sec^3\theta - Sec\theta\, d\theta$$
$$= tan\theta Sec\theta + \int sec\theta\, d\theta - \int Sec^3\theta\, d\theta$$
$$= tan\theta Sec\theta + \ln|Sec\theta + tan\theta| - \int Sec^3\theta\, d\theta$$
$$\int Sec^3 d\theta = \frac{1}{2} tan\theta Sec\theta + \frac{1}{2}\ln|Sec\theta tan\theta| + C$$

5. $\int \frac{xe^x}{(x+1)^2}$

$$u = xe^x, \qquad dv = \frac{1}{(x+1)^2}$$
$$du = (e^x + xe^x)dx, \qquad v = \frac{1}{x+1}$$
$$\int \frac{dv}{(x+1)^2} = \int \frac{dz}{z^2} = \int z^{-2}\, dz$$
$$\begin{matrix} z = x+1 \\ dz = dx \end{matrix} = \frac{z^{-2+1}}{-2+1} = \frac{z^{-1}}{-1} = \frac{-1}{z} = \frac{-1}{x+1}$$
$$= \frac{-xe^x}{x+1} + \int \left(\frac{-e^x}{x+1} - \frac{xe^x}{x+1}\right)dx = \frac{-xe^x}{x+1} + \int \frac{e^x}{x+1}dx + \int \frac{xe^x}{x+1}dx$$
$$u = xe^x$$
$$du = e^x(1+x)dx$$

$$\int \frac{xe^x}{(x+1)^2}dx = \frac{-xe^x}{x+1} - \int \left[\frac{-1}{x+1}\right][e^x(1+x)]dx = \frac{xe^x}{x+1} - \int -e^x dx$$
$$= \frac{-xe^x}{x+1} + \int e^x dx = \boldsymbol{\frac{-xe^x}{x+1} + e^x + C}$$

6. $\int_0^1 arcsenx\, dx = \int Sen^{-1}x\, dx$

$$u = Sen^{-1}x, \quad dv = dx$$
$$du = \frac{1}{\sqrt{1-x^2}}, \quad v = x$$
$$xSen^{-1}x - \int x.\frac{1}{\sqrt{1-x^2}}dx = xSen^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$$
$$z = 1 - x^2$$
$$\frac{dz}{2} = x\,dx$$
$$\int \frac{x\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\int \frac{dz}{z^{1/2}} = -\frac{1}{2}\int z^{-1/2}dz = -\frac{1}{2}\left[\frac{z^{-1/2+1}}{-1/2+1}\right] = \frac{-1}{2}\left[\frac{-z^{1/2}}{1/2}\right] = -z^{\frac{1}{2}}$$
$$= -(1-x^2)^{1/2}$$
$$\int Sen^{-1}x\,dx = xSen^{-1}x - \left[-\sqrt{1-x^2}\right] + C = \boldsymbol{xSen^{-1}x + \sqrt{1-x^2}}$$

SUSTITUCIONES TRIGONOMÉTRICAS

1. $\int \frac{dx}{4x^{2+1}} = \frac{1}{2}\int \frac{Sec^2\theta d\theta}{Sec\theta} = \frac{1}{2}\int Sec\theta d\theta = \frac{1}{2}[\ln|Sec\theta + g\theta] + C = \frac{1}{2}\left[\ln|\sqrt{4x^2+1} + 2x|\right] + C$

$$dx = \frac{1}{2}du, \quad x = \frac{1}{2}tan\theta$$

2. $\int \frac{dx}{x^2\sqrt{9-x^2}} \rightarrow \begin{matrix} a = 3 \\ u = x \end{matrix} \rightarrow \begin{matrix} x = 3Sen\theta, \quad x^2 = 9Sen^2\theta \\ \sqrt{9-x^2} = 3cos\theta, \quad dx = 3cos\theta d\theta \end{matrix}$

$$\int \frac{3Cos\theta d\theta}{9Sen^2\theta.3Cos\theta} = \frac{1}{9}\int \frac{d\theta}{Sen^2\theta} = \frac{1}{9}\int csc^2\theta d\theta = -\frac{1}{9}Cot\theta + C = -\frac{1}{9}\frac{Cos\theta}{Sen\theta} + C$$
$$= -\frac{1}{9}\left[\frac{\sqrt{9-x^2}}{3} \div \frac{x}{3}\right] + C = \boldsymbol{-\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C}$$

3. $\int \frac{dx}{(x^2+1)^{3/2}} \rightarrow \begin{matrix} x = tan\theta, \quad dx = Sec^2\theta d\theta \\ \sqrt{x^2+1} = 1Sec\theta \end{matrix}$

$$\int \frac{dx}{(\sqrt{x^2+1})^3} = \int \frac{Sec^2\theta d\theta}{(Sec\theta)^3} = \int \frac{d\theta}{Sec\theta} = \int Cos\theta\, d\theta = Sen\theta + C = \boldsymbol{\frac{x}{\sqrt{x^2+1}} + C}$$

4. $\int \frac{dx}{(x^2-4x)^{3/2}} = \int \frac{dx}{\left(\sqrt{x^2-4x}\right)^3} \to \int \frac{dx}{\left(\sqrt{(x-2)^2-2^2}\right)^3} \to$

$$a = 2,\ \ x - 2 = 2sec\theta,\ \ x = 2Sec\theta + 2$$

$$u = x - 2,\ \sqrt{(x-2)^2 - 2^2} = 2tan\theta,\ \ dx = 2Sec\theta tan\theta d\theta$$

$$\to \int \frac{2Sec\theta tan\theta d\theta}{(2tan\theta)^3} = \frac{2}{8}\int \frac{Sec\theta tan\theta}{tan^3\theta} d\theta = \frac{1}{4}\int \frac{Sec\theta}{tan^2\theta} d\theta = \frac{1}{4}\int \frac{Cos\theta}{Sen^2\theta} d\theta$$

$$\to z = Sen\theta,\ \ z^2 = Sen^2$$

$$dz = Cos\theta d\theta$$

$$\frac{1}{4}\int \frac{dz}{z^2} = \frac{1}{4}\int z^{-2}dz = \frac{1}{4}\left[\frac{z^{-1}}{-1}\right] = -\frac{1}{4z} = -\frac{1}{4sen\theta} + C = -\frac{1}{4\left[\frac{\sqrt{(x-2)^2-4}}{x-2}\right]} + C$$

$$= \frac{x-2}{4\sqrt{(x-2)^2-4}} + C$$

POTENCIAS DE SENOS Y COSENO

1. $\int Sen^5x\,dx = \int Sen^4xSenx\,dx = \int (Sen^2x)^2Senx\,dx = \int (1 - Cos^2x)^2Senx\,dx =$

$$\int (1 - 2Cos^2x + Cos^4x)Sen\,x\,dx = \int Senx\,dx - 2\int Cos^2\,xSenxdx + \int Cos^4xSenx\,dx$$

$$u = Cosx$$

$$du = -Senxdx$$

$$= -Cosx + 2\int u^2du - \int u^4du = -Cosx + \frac{2u^3}{3} - \frac{u^5}{5} + C \to \boldsymbol{-Cosx + \frac{2}{3}Cos^3x - \frac{1}{5}Cos^5 + C}$$

2. $\int Sen^3\,xCos^4xdx = \int Sen^2xSenxCos^4x\,dx = \int (1 - Cos^2x)Cos^4xSenx\,dx =$

$$\int Cos^4xSenx\,dx - \int Cos^6xSenx\,dx \to \begin{matrix} u = Cosx \\ du = -Senxdx \end{matrix}$$

$$\int Sen^3xCos^4xdx = -\int u^4du + \int u^6du = -\frac{u^5}{5} + \frac{u^7}{7} + C = -\boldsymbol{\frac{1}{5}Cos^5x} + \boldsymbol{\frac{1}{7}Cos^7}x + C$$

INTEGRALES DE SECANTES Y TANGENTES

1. $\int tan^5xdx = \int tan^2xtan^3\,xdx \to \int (Sec^2\,x - 1)tan^3xdx = \int Sec^2xtan^3xdx - \int tan^3x\,dx$

$$u = tanx$$

$$du = Sec^2xdx$$

$$\int tan^5x\,dx = \int u^3\,du - \left[\int tan^2xtanx\,dx\right] = \frac{u^4}{4} - \left[\int (Sec^2x - 1)tanxdx\right]$$

$$\frac{1}{4}tan^4x - \left[\int Sec^2xtanxdx - \int tanxdx\right] = \frac{1}{4}tan^4x - \int Sec^2xtanxdx + \int tanxdx$$

$$\frac{1}{4}tan^4x - \int udu + \ln|Secx| + C = \frac{1}{4}tan^4x - \frac{1}{2}u^2 + \ln|Secx| + C$$

$$\int tan^5x\,dx = \boldsymbol{\frac{1}{4}tan^4x - \frac{1}{2}tan^2x + \ln|Secx| + C}$$

2. $\int Sec^43xtan^33x\,dx \to \begin{matrix} u = tan3x \\ dx = \frac{du}{3Sec^23x} \end{matrix} \to \int Sec^43xu^3\frac{du}{3Sec^23x} = \frac{1}{3}\int Sec^2\,3x * u^3du$

$$\frac{1}{3}\int(1+tan^2 3x)\,u^3 = \frac{1}{3}(1+u^2)u^3 = \frac{1}{3}\int u^3+u^5\,du = \frac{1}{3}\left[\frac{1}{4}u^4+\frac{1}{6}u^6\right]+C$$

$$\frac{1}{3}\left[\frac{1}{4}tan^4 3x+\frac{1}{6}tan^6 3x\right]+C = \mathbf{\frac{1}{12}tan^4 3x+\frac{1}{18}tan^6 3x+C}$$

3. $\int\frac{tan^3x}{\sqrt{Secx}}\,dx = \int\frac{tan^2xtanx}{\sqrt{Secx}}\,dx = \int\frac{(Sec^2x-1)tanx}{(Secx)^{1/2}} = \int\frac{Sec^2xtanx-tanx}{(Secx)^{1/2}}\,dx =$

$$\int\frac{Sec^2xtanx}{Sec^{1/2}x}dx-\int\frac{tanx}{Sec^{1/2}x}dx \rightarrow \int Sec^{3/2}xtanxdx-\int tanx\,Sec^{-1/2}xdx$$

$$\int Secx * Sec^{1/2}xtanxde-\int tanxSec^{-1/2}x\,dx$$

$$=\int Sec^{1/2}xSec\,xtanxdx-\int Sec^{-3/2}x\,Secxtanxdx$$

$$u = Secx$$
$$du = Secxtanxdx$$

$$\int\frac{tan^3x}{\sqrt{Secx}}\,dx = \int u^{1/2}du-\int u^{-\frac{3}{2}}du = \frac{2}{3}u^{3/2}-(-2)u^{-\frac{1}{2}}$$

$$= \mathbf{\frac{2}{3}sec^{3/2}x+2Sec^{-1/2}+C}$$

FRACCIONES PARCIALES

1. $\frac{(3x-1)}{(x+2)(x-3)} = \frac{A}{(x+2)}+\frac{B}{(x-3)}(x+2)(x-3)$

$$3x-1 = A(x-3)+B(x+2)$$
$$3x-1 = Ax-3A+Bx+2B$$
$$3x-1 = Ax+Bx-3A+2B$$
$$3x-1 = x(A+B)-(3A+2B)$$

$$A = 3-\frac{8}{5},\quad B = \frac{7}{5} \rightarrow \frac{3x-1}{(x+2)(x+2)} = \frac{7/5}{x+2}+\frac{8/5}{x-3}$$

2. $\frac{5x^2+20x+6}{x^3-2x^2+x} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+1} = \frac{A(x+1)+B(x)+C(X)}{x(x+1)}$

$$\int\frac{5x^2+20x+6dx}{x(x+1)^2} = \int\frac{6}{x}dx-\int\frac{1}{x+1}dx+\int\frac{9}{(x+1)^2}dx$$

$$6\ln|x|-\ln|x+1|+9\left[\int\frac{dx}{(X+1)^2}\right] \rightarrow \begin{matrix} u = x+1 \\ du = dx \end{matrix}$$

$$= \mathbf{6\ln|x|-\ln|x+1|-\frac{9}{x+1}+C}$$

3. $\int\frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx = \frac{2}{x}-\frac{2}{x-1}+\frac{2x+4}{x^2+4} \rightarrow \begin{matrix} u = u^2+4 \\ \frac{du}{2} = xdx \end{matrix} =$

$$2\ln|x|-2\ln|x-1|+\int\frac{2x}{x^2+4}dx+4\int\frac{dx}{x^2+4} =$$

$$\mathbf{2\ ln|x| - 2\ ln|x-1| + ln\left|x^2+4\right| + \frac{4}{2} tan^{-1}\frac{x}{2} + C}$$

4. $\int \frac{8x^3+13x}{(x^2+2)^2}dx = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} = \frac{(Ax+B)(x^2+2)+(x+1)}{(x^2+2)^2}$